Fill Ups, True / False of Limits

Fill in the Blanks

Q. 1. $\text{Let } f(x) = \begin{cases} (x-1)^2 \sin\dfrac{1}{(x-1)} - |x| & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$

Be a real-valued function. Then the set of points where f(x) is not differentiable is......................

Ans. 0

Solution.

$$\text{Given } f(x) = \begin{cases} (x-1)^2 \sin\dfrac{1}{x-1} - |x|, x \neq 1 \\ -1 \ , \quad x = 1 \end{cases}$$

We know that | x | is not differentiable at x = 0

$\therefore\ (x-1)^2 \sin\dfrac{1}{x-1} - |x|$ Is not differentiable at x = 0.

At all other values of x, f(x) is differentiable.

∴ The req. set of points is {0}.

Q. 2. $\text{Let } f(x) = \begin{cases} \dfrac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$

If f(x) is continuous for all x, then k =..........

Ans. k = 7

Solution. It will be continuous at x = 2 if

$$\lim_{x\to2} f(x) = f(2) \Rightarrow \lim_{x\to2}\frac{x^3+x^2-16x+20}{(x-2)^2} = k$$

$$\Rightarrow \quad k = \lim_{x\to2}\frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x\to2}(x+5) = 7$$

$$\therefore \quad k = 7$$

Q. 3. A discontinuous function y =f(x) satisfying $x^2 + y^2 = 4$ is given by f(x) =......................

Ans. $f(x) = \sqrt{4-x^2},\ -2 \le x \le 0 = -\sqrt{4-x^2},\ 0 \le x \le 2$

Solution. $f(x) = \sqrt{4-x^2}, -2 \le x \le 0 \ = -\sqrt{4-x^2}, 0 \le x \le 2$

By choosing any arcs of circle $x^2 - y^2 = 4$, we can define a discontinuous function, one of which is

$$f(x) = \begin{cases} \sqrt{4-x^2}, & -2 \le x \le 0 \\ -\sqrt{4-x^2}, & 0 \le x \le 2 \end{cases}$$

Q. 4. $\lim_{x\to1}(1-x)\tan\frac{\pi x}{2} = ...$

Ans. $2/\pi$

Solution. KEY CONCEPT

(L' Hospital rule)

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if $\frac{f(a)}{g(a)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $0\times\infty$.

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} \qquad [\text{form } 0\times\infty]$$

$$=\lim_{x\to 1}\frac{1-x}{\cot(\pi x/2)} \qquad \left[\text{Form}\frac{0}{0}\right]$$

$$=\lim_{x\to 1}\frac{-1}{\frac{-\pi}{2}\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)} \quad [\text{Applying } L'\text{Hospital's rule}]$$

$$=\frac{2}{\pi}.$$

Q. 5. If f(x) = sin x, x ≠ np,n = 0, ± 1, ± 2,± 3, ... = 2, otherwise
and g(x) = x^2 + 1,x ≠ 0,2
= 4, x = 0
= 5, x = 2,
then $\lim_{x\to 0} g[f(x)]$ is ....

Ans. 1

Solution. Given that,

f (x) = sin x, x ≠ p,n = 0, ±1,± 2,... = 2, otherwise

And g(x) = x^2 + 1, x ≠ 0,2

= 4, x = 0 = 5, x = 2

Then $\lim_{x\to 0} g[f(x)]=\lim_{x\to 0} g(\sin x)\Rightarrow \lim_{x\to 0}(\sin^2 x+1)=1$

Q. 6. $$\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]=\text{.................}$$

Ans. -1

Solution.

$$\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$$

Let $L=\lim_{x\to-\infty}\frac{x^3}{1+|x|^3}\left[x\sin\left(\frac{1}{x}\right)+\frac{1}{x}\right]$

$$=\lim_{x\to-\infty}\frac{x^3}{|x|^3}\left[\frac{1}{1+\frac{1}{|x|x^2}}\right]\left[x\sin\left(\frac{1}{x}\right)+\frac{1}{x}\right]\ ...(1)$$

$$=\lim_{x\to-\infty}\frac{x^3}{|x|^3}.1=\lim_{x\to-\infty}\frac{x^3}{-x^3}=-1$$

Q. 7. If $f(9)=9,\ f'(9)=4$, then $\underset{x\to 9}{Lim}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ equals............

Ans. 4

Solution.

Given that $f(9)=9, f'(9)=4$

Then,

$$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(\sqrt{f(x)}-3)(\sqrt{f(x)}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{\sqrt{x}+3}{\sqrt{f(x)}+3}=\lim_{x\to 9}\frac{f(x)-9}{x-9}.\left[\frac{3+3}{3+3}\right]$$

$$=\lim_{x\to 9}\frac{f(x)-f(9)}{x-9}.1=f'(9)=4$$

Q. 8. ABC is an isosceles triangle inscribed in a circle of radius r. If AB = AC and h is the altitude from A to BC then the triangle ABC has

perimeter $P=2(\sqrt{2hr-h^2})+\sqrt{2hr})$ and area $A=$.............. also $\lim_{h\to 0}\frac{A}{P^3}=$........................

Ans. $\sqrt{2rh-h^2},\ \frac{1}{128r}$

Solution. In Δ ABC, AB = AC

AD ⊥ BC (D is mid pt of BC)

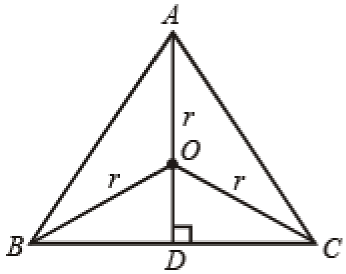
Let r = radius of circumcircle

∴ OA = OB = OC = r

$$\text{Now } BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2}$$

$$= \sqrt{2rh - h^2}$$

$$\therefore \quad BC = 2\sqrt{2rh - h^2}$$



$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2} \times BC \times AD = h\sqrt{2rh - h^2}$$

$$\text{Also } \lim_{h\to 0} \frac{A}{p^3} = \lim_{h\to 0} \frac{h\sqrt{2rh-h^2}}{8(\sqrt{2rh-h^2} + \sqrt{2rh})^3}$$

$$= \lim_{h\to 0} \frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}(\sqrt{2r-h}) + \sqrt{2r})^3}$$

$$= \lim_{h\to 0} \frac{\sqrt{2r-h}}{8(2r-h+\sqrt{2r})^3}$$

$$= \frac{\sqrt{2r}}{8(\sqrt{2r}+\sqrt{2r})^3} = \frac{\sqrt{2r}}{8.8.2r.\sqrt{2r}} = \frac{1}{128r}$$

Q. 9. $\underset{x\to\infty}{Lt}\left(\frac{x+6}{x+1}\right)^{x+4} =$ ....

Ans. e^5

Solution.

$$\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x\to\infty}\left\{\left[1+\frac{5}{x+1}\right]^{\frac{x+1}{5}}\right\}^{5\left(\frac{x+4}{x+1}\right)}$$

[Using $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$]

$$e^{\lim_{x\to\infty} 5\left(\frac{x+4}{x+1}\right)} = e^{5\lim_{x\to\infty}\left(\frac{1+4/x}{1+1/x}\right)} = e^5$$

Q. 10. Let f(x) = x | x |. The set of points where f(x) is twice differentiable is ....

Ans. R – {0}

Solution. We have,

$$f(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

$$f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

Clearly f '' (x) exists at every pt. except at x = 0

Thus f (x) is twice differentiable on R – {0}.

Q. 11. Let $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$, where $[\bullet]$ **denotes the greatest integer function. The domain of f is... and the points of discontinuity of f in the domain are.....**

Ans. (-∞, - 1) ∪ [0, ∞), I – {0} where I is the set of integer except n = –1

Solution. Thus function is not defined for those values of x for which [x + 1] = 0.

In other words it means that

$0 \leq x + 1 < 1$ or $-1 \leq x < 0$ ....(1)

Hence the function is defined outside the region given by (1).

Required domain is] - ∞, -1[∪ [0, ∞] Now, consider integral values of x say x = n

$$\text{R.H.L.} = \lim_{h\to 0} [n+h]\sin\frac{\pi}{[n+1+h]} = n\sin\frac{\pi}{(n+1)}$$

$$\text{L.H.L.} = \lim_{h\to 0}[n-h]\sin\frac{\pi}{[n+1-h]} = (n-1)\frac{\pi}{n}$$

Clearly RHL ≠ LHL. Hence the given function is not continuous for integral values of n

(n ≠ 0, –1).

At x = 0, f (0) = 0,

$$\lim f(0+h) = \lim_{h\to 0}[h]\sin\frac{\pi}{[h+1]} = 0$$

The function is not defined for x < 0. Hence we cannot find lim f (0 – h). Thus f (x) is continuous at x = 0. Hence the points of discontinuity are given by I – {0} where I is set of integers n except n = –1

Q. 12. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = \ldots$

Ans. e^2

Solution. KEY CONCEPT

$$\lim_{x\to 0}[f(x)]^{g(x)} = e^{\lim_{x\to 0} g(x)\log f(x)}$$

$$\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = e^{\lim_{x\to 0}\frac{1}{x^2}\log\left[\frac{1+5x^2}{1+3x^2}\right]}$$

$$= e^{\lim_{x\to 0}\left[5.\frac{\log(1+5x^2)}{5x^2} - 3.\frac{\log(1+3x^2)}{3x^2}\right]}$$

$$= e^{5-3} = e^2$$

Q. 13. Let f(x) be a continuous function defined for 1 ≤ x ≤ 3. If f(x) takes rational values for all x and f (2) = 10, then f(1.5) =............

Ans. 10

Solution. Since f (x) is given continuous on the closed bounded interval [1, 3], f (x) is bounded and assumes all the values lying in the interval [m, M] where

m = min f (x) and M = max f (x)

$1 \leq x \leq 3 \Rightarrow f(1) \leq f(x) \leq (3)$

If $m \gtrless M$, then $f(x)$ Must assume all the irrational values lying in the [m, M]. But since f (x) takes only rational values, we must have m = M i.e., f (x) must be a constant function.

As f (2) = 10, we get

$f(x) = 10 \ \forall x \in [1, 3] \Rightarrow f(1.5) = 10$

True / False

Q. 1. If $\underset{x \to a}{Lt} [f(x) g(x)]$ **exists then both** $\underset{x \to a}{Lt} f(x)$ **and** $\underset{x \to a}{Lt} g(x)$ exist.

Ans. F

Solution.

Consider $f(x) = \frac{|x-a|}{x-a}, g(x) = \frac{x-a}{|x-a|}$

Then $\lim_{x \to a} (f(x) g(x))$ exists but neither $\lim_{x \to a} f(x)$

$\lim_{x \to a} g(x)$ exists.

Subjective Questions of Limits

Q. 1. Evaluate $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}, (a \neq 0)$

Ans. $\frac{2}{3\sqrt{3}}$

Solution.

$$\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$$

$$= \lim_{x \to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{3(a-x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(\sqrt{3a+x}+2\sqrt{x})}{3(\sqrt{a+2x}+\sqrt{3x})} = \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

NOTE: The given limit is of the form 0/0. Hence limit of the function can also be find out by using L' Hospital's Rule

Q. 2. f (x) is the integral of $\frac{2\sin x - \sin 2x}{x^3}, x \neq 0$, find $\lim_{x \to 0} f'(x)$

Ans. 1

Solution.

$$f(x)=\int \frac{2\sin x-\sin 2x}{x^3}dx,\ x\neq 0$$

$$\therefore\quad f'(x)=\frac{2\sin x-\sin 2x}{x^3},\ x\neq 0$$

$$\therefore\quad \lim_{x\to 0} f'(x)=\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$$

$$=\lim_{x\to 0}\frac{2\sin x(1-\cos x)(1+\cos x)}{x^3(1+\cos x)}$$

$$=\lim_{x\to 0} 2.\frac{\sin^3 x}{x^3}.\frac{1}{1+\cos x}$$

$$=2\times(1)^3\times\frac{1}{2}=1$$

Q. 3. Evaluate : $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

Ans. $a^2\cos a + 2a\sin a$

Solution.

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]+2ah\sin(a+h)+h^2\sin(a+h)}{h}$$

$$=\lim_{h\to 0}\frac{a^2\left[2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}\right]}{2\times\frac{h}{2}}+2a\sin(a+h)$$

$$+h\sin(a+h)$$

$= a^2\cos a + 2a\sin a$

Q. 4. Let f (x + y) = f (x) + f (y) for all x and y. If the function f(x) is continuous at x = 0, then show that f (x) is continuous at all x.

Solution. As f (x) is continuous at x = 0, we have

$$LHL = RHL = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow f(0) + \lim_{h\to 0} f(-h) = f(0) + \lim_{h\to 0} f(h) = f(0)$$

[Using the given property $f(x+y) = f(x) + f(y)$]

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 0 \qquad \ldots(1)$$

Now let x = a be any arbitrary point then at x = a,

$$LHL = \lim_{h\to 0} f(a-h) = \lim_{h\to 0}[f(a) + f(-h)]$$

[Using, $f(x+y) = f(x) + f(y)$]

$$= f(a) + \lim_{h\to 0} f(-h) = f(a) \qquad [\text{using eq}^n (1)]$$

Similarly, R.H.L. $= \lim_{h\to a} f(a+h) = f(a)$

Thus, we get

$$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a)$$

$\Rightarrow$ f is continuous at x = a. But a is any arbitrary point

$\therefore$ f is continuous $\forall x \in R$.

Q. 5. Use the formula $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$ to find $\lim_{x\to 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$

Ans. 2 ln 2

Solution.

$$\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} = \lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{x\to 0} \frac{(2^x - 1)(\sqrt{1+x}+1)}{1+x-1}$$

$$= \lim_{x\to 0} \frac{2^x - 1}{x} . \lim_{x\to 0}(\sqrt{1+x}+1)$$

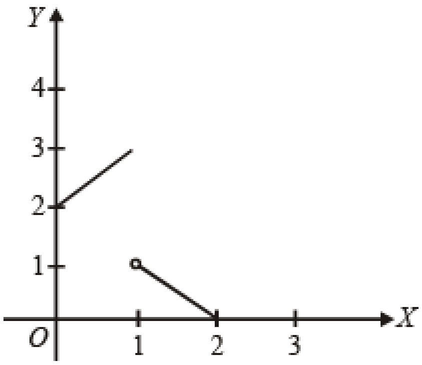
$$= \ln 2 . (1+1) = 2 \ln 2.$$

Q. 6. Let $f(x) = \begin{cases} 1+x, 0 \le x \le 2 \\ 3-x, 2 \le x \le 3 \end{cases}$

Determine the form of g(x) = f [f(x) and hence find the points of discontinuity of g, if any

Ans. $g(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$ discontinuity at $x=1, 2$

Solution. Graph of f (f (x)) is



Clearly form graph f (f (x)) is discontinuous at x = 1 and 2.

Q. 7. Let $f(x)=\begin{cases}\dfrac{x^2}{2} & ,0\le x<1\\ 2x^2-3x+\dfrac{3}{2}, & 1\le x\le 2\end{cases}$

Discuss the continuity of f, f ' and f " on [0, 2].

Ans. f and f' are continuous and f" is discontinuous on [0, 2]

Solution. We have $f(x)=\dfrac{x^2}{2}, 0\le x<1 = 2x^2-3x+\dfrac{3}{2}, 1\le x\le 2$

Here f (x) is continuous everywhere except possibly at x = 1

$\Rightarrow$ At $x=1$, $Lf'=\dfrac{2}{2}\times 1=1$; $Rf'=4\times 1-3=1$

$\Rightarrow$ f is differentiable and hence continuous at x = 1

$\therefore$ f (x) is continuous on [0, 2]

$$f'(x) = x,\ 0 \le x < 1$$
$$= 4x - 3,\ 1 \le x \le 2$$

At $x = 1$,

$$\lim_{x \to 1^-} f'(x) = \lim_{h \to 0} f'(1-h) = \lim_{h \to 0} (1-h) = 1$$
$$\lim_{x \to 1^+} f'(x) = \lim_{h \to 0} f'(1+h) = \lim_{h \to 0} 4(1+h) - 3 = 1$$
$$f'(1) = 4 - 3 = 1$$
$\therefore$ f' is continuous at $x = 1$

$\therefore$ f' is continuous on [0, 2]

$$f''(x) = \begin{cases} 2 & , 0 \le x < 1 \\ 4 & , 1 \le x \le 2 \end{cases}$$

Clearly $f''(x)$ is discontinuous at $x = 1$,

$\therefore$ $f''(x)$ is discontinuous on [0, 2].

Q. 8. Let f (x) = x^3 - x^2 + x + 1 and

$$g(x) = \max\{f(t); 0 \le t \le x\},\ 0 \le x \le 1$$
$$= 3 - x \qquad 0 \le x \le 2$$

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

Ans. cont. on (0, 2) and differentiable on (0, 2) – {1}

Solution. Given f (x) = $x^3 - x^2 + x + 1$

$$\therefore \quad f'(x) = 3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right)$$
$$= 3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{1}{3}\right]$$
$$= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right] > 0 \ \forall\, x \in R.$$

Hence f (x) is an increasing function of x for all real values of x.

Now max $[f(t): 0 \le t \le x]$ means the greatest value of $f(t) \text{ in } 0 \le t \le x$ which is obtained at t = x, since f (t) is increasing for all t.

$$\therefore \quad \max.\ [f(t): 0 \le t \le x] = x^3 - x^2 + x + 1$$

Hence the function g is defined as follows :

$$g(x) = x^3 - x^2 + x + 1 \quad \text{when } 0 \le x \le 1$$
$$= 3 - x \quad \text{when } 1 < x \le 2$$

Now it is sufficient to discuss the continuity and differentiability of g (x) at x = 1. Since for all other values of x, g (x) is clearly continuous and differentiable, being a polynomial function of x.

We have, g (1) = 2

$$g(1-0) = \lim_{h\to 0} [(1-h)^3 - (1-h)^2 + (1-h) + 1] = 2$$
$$g(1+0) = \lim_{h\to 0} [3-(1+h)] = 2$$

Hence g (x) is continuous at x = 1 Now,

$$Lg'(1) = \lim_{h\to 0} \frac{[(1-h)^3 - (1-h)^2 + (1-h) + 1] - 2}{-h}$$

$$= \lim_{h\to 0} \frac{1 - 3h + 3h^2 - h^3 - 1 + 2h - h^2 + 1 - h + 1 - 2}{-h}$$

$$= \lim_{h\to 0} \frac{-2h + 2h^2 - h^3}{-h} = \lim_{h\to 0} [2 - 2h + h^2] = 2$$

$$Rg'(1) = \lim_{h\to 0} \frac{[3-(1+h)-2]}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

Since $Lg'(1) \ne Rg'(1)$, the function g (x) is not differentiable at x = 1 Hence g (x) is continuous on (0, 2). It is also differentiable on (0, 2) except at x = 1.

Q.9. Let f (x) be defined in the interval [–2, 2] such that

$$f(x) = \begin{cases} -1, -2 \le x \le 0 \\ x - 1, 0 < x \le 2 \end{cases}$$

and g(x) = f (|x |) + |f (x)|

Test the differentiability of g(x) in (– 2, 2).

Ans. not differentiable at x = 1

Solution.

$$\text{We have } f(x) = -1,\ -2 \le x \le 0$$
$$= x-1,\ 0 < x \le 2$$
$$\text{and } g(x) = f(|x| + |f(x)|$$

Hence g(x) involves | x | and | x –1| or | – 1| = 1

Therefore we should divide the given interval (–2, 2) into the following intervals.

I_1	I_2	I_3
$[-2, 2] = [-2, 0)$	$[0, 1)$	$[1, 2]$
$x = -ve$	$+ ve$	$+ ve$
$\|x\| = -x$	x	x
$f(x) = -1$	$x-1$	$x-1$
$f(\|x\|) = -1$	$= x-1$	$= x-1$
$\|f(x)\| = \|-1\|$	$\|x-1\|$	$\|x-1\|$
$= 1$	$= -(x-1)$	$= x-1$

$\therefore$ Using above we get

$$g(x) = f|x| + |f(x)|$$
$$= -1 + 1 = 0 \text{ in } I_1$$
$$= x-1 - (x-1) = 0 \text{ in } I_2$$
$$= x-1 + x-1 = 2(x-1) \text{ in } I_3$$

Hence g (x) is defined as follows :

$$g(x) = \begin{cases} 0, & -2 \le x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

$Lg'(1) = 0; Rg'(1) = 2$ (not equal)

Hence $g(x)$ is not differentiable at $x = 1$.

Q.10. Let f (x) be a continuous an d g (x) be a discontinuous function. prove that f (x) + g (x) is a discontinuous function.

Solution. Let h (x) = f (x) + g (x) be continuous.

Then, g (x) = h (x) – f (x)

Now, h (x) and f (x) both are continuous functions.

∴ h (x) – f (x) must also be continuous. But it is a contradiction as given that g (x) is discontinuous.

Therefore our assumption that f (x) + g (x) a continuous function is wrong and hence f (x) + g (x) is discontinuous.

Q.11. Let f (x) be a function satisfying the condition f (– x)= f (x) for all real x. If f ' (0) exists, find its value.

Ans. f ' (0) = 0

Solution. Given that f (x) is a function satisfying

$$f(-x) = f(x), \forall x \in R \qquad ...(1)$$

Also $f'(0)$ exists

$$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$$

Now, $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) \qquad ...(2)$$

Again $Lf'(0) = f'(0)$

$$\Rightarrow \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) \qquad ...(3)$$

[Using eq. (1)]

From equations (2) and (3), we get

⇒ f ' (0) = 0

Q.12. Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & \pi/2 < x \le \pi \end{cases}$$

Is continuous for $0 \le x \le \pi$.

Ans.

$a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$

Solution. Given that,

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & , \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & , \frac{\pi}{2} < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

∴ f (x) must be continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

$$\lim_{x \to \left(\frac{\pi}{4}\right)^-} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \lim_{h \to 0} f\left(\frac{\pi}{4} - h\right) = \frac{2\pi}{4}\cot\frac{\pi}{4} + b$$

$$\Rightarrow \lim_{h \to 0}\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right) = \frac{\pi}{2} + b$$

$$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b$$

$$\Rightarrow a - b = \frac{\pi}{4} \qquad \text{...(1)}$$

Also, $\lim_{x \to \left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right) = 2.\frac{\pi}{2}\cot\frac{\pi}{2} + b$$

$$\Rightarrow \lim_{h \to 0} a\cos 2\left(\frac{\pi}{2} + h\right) - b\sin\left(\frac{\pi}{2} + h\right) = b$$

$$\Rightarrow a\cos\pi - b\sin\frac{\pi}{2} = b \Rightarrow -a - b = b$$

$$\Rightarrow a + 2b = 0 \qquad \text{...(2)}$$

Solving (1) and (2), we get $a = \frac{\pi}{6}$ and $b = \frac{-\pi}{12}$.

Q.13. Draw a graph of the function y = [x] + | 1 – x |, -1 ≤ x ≤ 3 . Determine the points, if any, where this function is not differentiable.

Ans. x = 0, 1, 2, 3

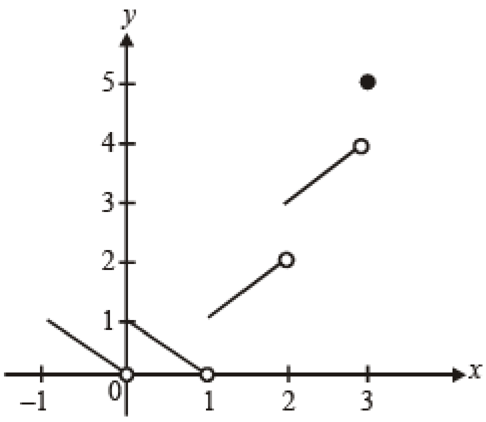
Solution. We have, [x] + | 1– x |, -1 ≤ x ≤ 3

NOTE THIS STEP

$$\text{or } y = \begin{cases} -1+1-x & , -1 \le x < 0 \\ 0+1-x & , 0 \le x < 1 \\ 1-1+x & , 1 \le x < 2 \\ 2-1+x & , 2 \le x < 3 \\ 3-1+x & , x = 3 \end{cases}$$

$$\text{or } \quad y = \begin{cases} -x \ , & -1 \le x < 0 \\ 1-x \ , & 0 \le x < 1 \\ x \ , & 1 \le x < 2 \\ 1+x \ , & 2 \le x < 3 \\ 2+x \ , & x = 3 \end{cases}$$

From graph we can say that given functions is not differentiable at x = 0, 1, 2, 3.



Q.14. $$\text{Let } f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x > 0 \end{cases}$$

Determine the value of a, if possible, so that the function is continuous at x = 0

Ans. a = 8

Solution. We are given that,

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & ,x<0\\ a & ,x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ,x>0\end{cases}$$

Here L.H.L at $(x=0)$

$$=\lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}=\lim_{h\to 0}\frac{1-\cos 4h}{h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 2h}{4h^2}.4=8$$

R.H.L at $(x=0)$

$$=\lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}=\lim_{h\to 0}\frac{\sqrt{h}(\sqrt{16+h}+4)}{16+\sqrt{h}-16}$$

$$=\lim_{h\to 0}\sqrt{16+\sqrt{h}}+4=\sqrt{16}+4=8$$

For continuity of function $f(x)$, we must have

L.H.L. = R.H.L. = f(0)

$\Rightarrow\ f(0)=8 \Rightarrow\ a=8$

Q.15. A function f: R → R satisfies the equation f (x + y) = f (x) f (y) for all x, y in R and f (x) ≠ 0 for any x in R. Let the function be differentiable at x = 0 and f' (0) = 2. Show that f' (x) = 2 f (x) for all x in R. Hence, determine f (x).

Ans. f (x) = e^{2x}

Solution. We are given

$f(x+y)=f(x)\,f(y),\ \forall\ x,y\in R$

f (x) ≠ 0, for any x

f is differentiable at x = 0, f ' (0) = 2

To prove that $f'(x)=2f(x),\ \forall\ x\in R$ and to find $f(x)$.

We have for x = y = 0

f (0 + 0) = f (0) f (0)

⇒ f (0) = [f (0)]2 ⇒ f (0) = 1

Again $f'(0) \neq 2$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=2 \Rightarrow \lim_{h\to 0}\frac{f(0)f(h)-f(0)}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0)[f(h)-1]}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-1}{h}=2 \quad ...(1) \quad [\text{Using } f(0)=1]$$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} = \lim_{h\to 0} f(x)\left(\frac{f(h)-1}{h}\right)$$

$$=f(x)\lim_{h\to 0}\left[\frac{f(h)-1}{h}\right]$$

$$=f(x).\,2 \quad [\text{Using eq. (1)}]$$

$$=2f(x)$$

$$\text{Also, } \frac{f'(x)}{f(x)}=2$$

Integrating on both sides with respect to x, we get

$$\log|f(x)| = 2x + C$$

$$\text{At } x=0,\ \log f(0)=C \Rightarrow C=\log 1=0$$

$$\therefore \log|f(x)|=2x \Rightarrow f(x)=e^{2x}$$

Q.16. Find $\lim_{x\to 0}\{\tan(\pi/4+x)\}^{1/x}$

Ans. e^2

Solution.

$$\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}\right)+x\right\}^{\frac{1}{x}} = e^{\lim_{x\to 0}\log\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}}$$

$$\left[\text{Using } \lim_{x\to a} f(x)=e^{\lim_{x\to 0}\log f(x)}\right]$$

$$= e^{\lim_{x\to 0}\frac{\log\tan\left(\frac{\pi}{4}+x\right)}{x}} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= e^{\lim_{x\to 0}\left[\frac{\sec^2\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}\right)+x}\right]} \qquad \text{[Using } L' \text{ Hospital's rule]}$$

$$= e^{\frac{2}{1}} = e^2$$

Q.17.

$$\text{Let } f(x) = \begin{cases} \{1+|\sin x|\}^{a/|\sin x|} & ; \quad \frac{\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ e^{\tan 2x/\tan 3x} & ; \quad 0 < x < \frac{\pi}{6} \end{cases}$$

Determine a and b such that f(x) is continuous at x = 0

Ans. $a = \frac{2}{3}, \; b = e^{2/3}$

Solution.

$$\text{Given that, } f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & \frac{-\pi}{6} < x < 0 \\ b & x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & 0 < x < \frac{\pi}{6} \end{cases}$$

is continuous at $x = 0$

$\therefore \quad \lim_{h\to 0} f(0-h) = f(0) = \lim_{h\to 0} f(0+h)$

We have,

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0}[1+|\sin(-h)|]^{\frac{a}{|\sin(-h)|}}$$

$$= \lim_{h\to 0}[1+\sin h]^{\frac{a}{\sin h}}$$

$$e^{\lim_{h\to 0}\frac{a}{\sin h}\log(1+\sin h)} = e^a$$

and $f(0) = b$

$\therefore \quad e^a = b \qquad \ldots(1)$

Also $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} e^{\frac{\tan 2h}{\tan 3h}}$

$$= e^{\lim_{h\to 0} \frac{\tan 2h}{2h} \times \frac{3h}{\tan 3h} \times \frac{2}{3}} = e^{\frac{2}{3}}$$

$\therefore \quad e^{\frac{2}{3}} = b$ ...(2)

From (1) and (2)

$$e^a = b = e^{\frac{2}{3}} \Rightarrow a = \frac{2}{3} \text{ and } b = e^{\frac{2}{3}}$$

Q.18. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real x and y. If f '(0) exists and equals – 1 and f (0) =1, find f (2).

Ans. f(2) = –1

Solution.

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \text{...(1)}$$

Putting y = 0 and f (0) = 1 in (1), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{2}[f(x)+1]$$

$$\therefore \quad f(x) = 2f\left(\frac{x}{2}\right) - 1 \quad \text{... (2)}$$

Now, $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{f(2x)+f(2h)}{2} - f(x)\right], \text{ by (1)}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{(2f(x)-1)+(2f(h)-1)}{2} - f(x)\right], \text{ by (2)}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}[f(h)-1]$$

$$= \underset{h\to 0}{Lt} \frac{f(h)-f(0)}{h} = f'(0) = -1$$

Hence f '(x) = – 1, integrating, we get

f (x) = – x + c. Putting x = 0, we get

f (0) = c = 1 by (1) ∴ f (x) = 1 – x

f (2) = 1 – 2 = – 1

Q.19. Determine the values of x for which the following function fails to be continuous or differentiable:

$$f(x)=\begin{cases}1-x, & x<1\\(1-x)(2-x), & 1\le x\le 2\\3-x, & x>2\end{cases}$$ Justify your answer..

Ans. f is continuous and differentiable at all points except at x = 2.

Solution. By the given definition it is clear that the function f is continuous and differentiable at all points except possible at x = 1 and x = 2.

Continuity at x = 1

$$\text{L.H.L.}=\lim_{h\to 0}[1-(1-h)]=\lim_{h\to 0}h=0$$

$$\text{R.H.L.}=\lim_{h\to 0}[1-(1+h)][2-(1+h)]$$

$$=\lim_{h\to 0}\{-h(1-h)\}=0$$

$$\text{Also, } f(1)=0$$

$$\therefore\quad \text{L.H.L.}=\text{R.H.L.}=f(1)=0$$

Therefore, f is continuous at x = 1

Now, differentiability at x = 1

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}, h > 0$$
$$= \lim_{h \to 0} \frac{(1-(1-h)-0}{-h} = \lim_{h \to 0} \left(\frac{h}{-h}\right) = -1$$
$$\text{and } Rf'(1) = \lim_{h \to 0} \frac{f((1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{\{1-(1-h)\}\{2-(1-h)\}-0}{h}$$
$$= \lim_{h \to 0} \frac{-h(1-h)}{h} = \lim_{h \to 0} (h-1) = -1$$

Since Lf ' (1) = Rf ' (1)

Hence, f is differentiable at x = 1

Continuous at x = 2

$$\text{L.H.L.} = \lim_{h \to 0} [1-(2-h)][2-(2-h)]$$
$$\lim_{h \to 0} \{(-1+h)\}\{h\} = 0$$
$$\text{and R.H.L.} = \lim_{h \to 0} [3-(2+h)] = \lim_{h \to 0} (1-h) = 1$$

Since L.H.L.$\neq$ R.H.L., therefore f is not continuous at x = 2. As such f cannot be differentiable at x = 2. Hence f is continuous and differentiable at all points except at x = 2.

Q. 20. Let f (x), x $\geq$ 0, be a non-negative continuous function,

and let $F(x) = \int_0^x f(t)\,dt, x \geq 0$. If for some $c > 0, f(x) \leq cF(x)$ **for all x $\geq$ 0, then show that f(x) = 0 for all x $\geq$ 0.**

Solution. Given that,

$$F(x) = \int_0^x f(t)dt$$

NOTE THIS STEP

$\therefore$ F ' (x) = f (x).1– f (0).0

[Using Leibnitz theorem]

$\Rightarrow \quad F'(x) = f(x)$ ... (1), $\forall\, x \geq 0$

Also $F(0) = \int_0^0 f(t)dt = 0$

But given that $f(x) \leq cF(x),\ \forall\, x \geq 0$

$\therefore$ We get $f(0) \leq cF(0) = 0$

$\therefore \quad f(0) \leq 0$ ....(2)

But ATQ f (x) is non- negative continuous function on [0, ∞)

$\therefore \quad f(x) \geq 0$

$\therefore \quad f(0) \geq 0$ ....(3)

$\therefore$ From (2) and (3) $f(0) = 0$

Again $f(x) \leq cF(x) \forall\, x \geq 0$, we get

$f(x) - cF(x) \leq 0$

$\Rightarrow \quad F'(x) - cF(x) \leq 0, \forall\, x \geq 0$ [Using equation (1)]

$e^{cx}F'(x) - ce^{-cx}F(x) \leq 0$

[Multiplying both sides by e^{-cx} (I.F.) and keeping in

mind that $e^{-cx} > 0, \forall\, x$]

$\Rightarrow \quad \frac{d}{dx}[e^{-cx}F(x)] \leq 0$

$\Rightarrow \quad g(x) = e^{-cx}F(x)$ is a decreasing function on [0,∞).

That is $g(x) \le g(0)$ for all $x \ge 0$

But $g(0) = F(0) = 0$

$\therefore \quad g(x) \le 0, \forall x \ge 0$

$\Rightarrow \quad e^{-cx} F(x) \le 0, \forall x \ge 0$

$\Rightarrow \quad F(x) \le 0, \forall x \ge 0$

$\therefore \quad f(x) \le cF(x) \le 0, \forall x \ge 0$

$[\therefore c > 0$ and using $f(x) \le cF(x)]$

$\Rightarrow \quad f(x) \le 0, \forall x \ge 0$

But given $f(x) \ge 0$

$\Rightarrow \quad f(x) = 0, \forall x \ge 0.$

Q. 21. Let a ∈ R. Prove that a function f : R → R is differentiable at a if and only if there is a function g : R → R which is continuous at a and satisfies f(x) - f(a) = g(x) (x - a) for all x ∈ R.

Solution. (I) g is continuous at α and

$f(x) - f(\alpha) = g(x)(x - \alpha), \forall x \in R$

$\Rightarrow$ Since g is continuous at $x = \alpha$

and $g(x) = \dfrac{f(x) - f(\alpha)}{x - \alpha}$

We should have, $\lim\limits_{x \to \alpha} g(x) = g(\alpha)$

$$\Rightarrow \quad \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = g(\alpha) \quad \Rightarrow \quad f'(\alpha) = g(\alpha)$$

(II) $\because \quad f(x)$ is differentiable at $x = \alpha$

$$\therefore \quad \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha)$$

exists and is finite.

Let us define,

$$g(x) = \begin{cases} \dfrac{f(x) - f(\alpha)}{x - \alpha}, & x \ne \alpha \\ f'(\alpha), & x = \alpha \end{cases}$$

Then, $f(x) - f(\alpha) = (x - \alpha)\, g(x), \ \forall \ x \ne \alpha.$

Now for continuity of $g(x)$ at $x = \alpha$

$$\lim_{x \to \alpha} g(x) = \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha) = g(\alpha)$$

$\therefore \quad g$ is continuous at $x = \alpha$.

Q. 22. Let $f(x)=\begin{cases}x+a & \text{if } x<0\\ |x-1| & \text{if } x\geq 0,\end{cases}$ and

$g(x)=\begin{cases}x+1 & \text{if } x<0\\ (x-1)^2+b & \text{if } x\geq 0,\end{cases}$ **Where a and b are non-negative real numbers.**

Determine the composite function g o f. If (g o f) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is g o f differentiable at x = 0?

Ans. $g(f(x))=\begin{cases}x+a+1. & \text{if } x<-a\\ (x+a-1)^2+b & \text{if } a\leq x<0\\ x^2+b & \text{if } 0\leq x\leq 1\\ (x-2)^2+b & \text{if } x>1\end{cases}$, $a=1, b=0$, gof is differentiable at $x=0$

Solution. Given that

$$f(x)=\begin{cases}x+a, & \text{if } x<0\\ |x-1|, & \text{if } x\geq 0\end{cases}=\begin{cases}x+a, & \text{if } x<0\\ 1-x, & \text{if } 0\leq x<1\\ x-1, & \text{if } x\geq 1\end{cases}$$

and $g(x)=\begin{cases}(x+1), & \text{if } x<0\\ (x-1)^2+b, & \text{if } x\geq 0\end{cases}$

where $a,b\geq 0$

Then $(gof)(x)=g[f(x)]$

NOTE THIS STEP

$$=\begin{cases}f(x)+1, & \text{if } f(x)<0\\ [f(x)-1]^2+b, & \text{if } f(x)\geq 0\end{cases}$$

(Using definition of g(x)

Now, f (x) < 0 when x + a < 0 i.e. x < – a

f (x) = 0 when x = – a or x = 1

f (x) > 0 when – a < x < 1 or x > 1

$$g(f(x)) = \begin{cases} f(x)+1, & \text{if } x < -a \\ [f(x)-1]^2 + b, & \text{if } x = -a \text{ or } x = 1 \\ [f(x)-1]^2 + b, & \text{if } -a < x < 0 \\ [f(x)-1]^2 + b, & \text{if } 0 \le x < 1 \\ [f(x)-1]^2 + b, & \text{if } x > 1 \end{cases}$$

[Keeping in mind that x = 0 and 1 are also the breaking points because of definition of f (x)]

$$\therefore g[f(x)] = \begin{cases} x+a+1, & \text{if } x < -a \\ (x+a-1)^2 + b, & \text{if } -a \le x < 0 \\ (1+x)-1)^2 + b, & \text{if } 0 \le x \le 1 \\ (x-1-1)^2 + b, & \text{if } x > 1 \end{cases}$$

Substituting the value of f (x) under different conditions).

$$\therefore \quad g[f(x)] = \begin{cases} x+a+1, & \text{if } x < -a \\ (x+a-1)^2 + b, & \text{if } -a \le x < 0 = F(x)(\text{say}) \\ x^2 + b, & \text{if } 0 \le x \le 1 \\ (x-2)^2 + b, & \text{if } x > 1 \end{cases}$$

Now given that g of (x) ≡ F (x) is continuous for all real numbers, therefore it will be continuous at – a

⇒ L.H.S = R.H.L = f (– a)

$$\lim_{h \to 0} F(-a-h) = \lim_{h \to 0} F(-a+h) = f(-a)$$

$$\text{Now, } \lim_{h \to 0} F(-a-h) = \lim_{h \to 0} (-a-h+a+1) = 1$$

$$\lim_{h \to 0} F(-a+h) = \lim_{h \to 0} (-a+h+a-1)^2 + b = 1+b$$

$$F(-a) = 1+b$$

Thus we should have 1 = 1 + b ⇒ b = 0.

Again for continuity at x = 0

$$\text{L.H.L} = f(0)$$

$$\Rightarrow \lim_{h \to 0} f(0-h) = f(0)$$

$$\Rightarrow \lim_{h \to 0} f(-h+a-1)^2 + b = b \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1$$

For a = 1 and b = 0, g of becomes

$$gof(x) = \begin{cases} x+2, & x < -1 \\ x^2, & -1 \le x \le 1 \\ (x-2)^2 & x > 1 \end{cases}$$

Now to check differentiability of g of (x) at x = 0 We see, g of (x) = x^2 = F(x)

⇒ F ' (x) = 2x which exists clearly at x = 0. G of is differentiable at x = 0

Q. 23. If a f unction f : [-2a, 2a]→R is an odd function such that f(x) = f(2a – x) for x ∈[a, 2a] and the left hand derivative at x = a is 0 then find the left hand derivative at x = – a.

Ans. 0

Solution. Given that f : [– 2a, 2a] → R

f is an odd function.

Lf ' at x = a is 0.

$$\Rightarrow \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = 0 \qquad \text{...(1)}$$

To find Lf ' at x = – a which is given by

$$\lim_{h\to 0}\frac{f(-a-h)-f(-a)}{-h}=\lim_{h\to 0}\frac{-f(a+h)+f(a)}{-h}$$

$$[\because f(-x)=-f(x)]$$

$$=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Substiting this values in last expression we get

$$Lf'(-a)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{h}$$

$$=0 \qquad \text{[Using equation (1)]}$$

Hence $Lf'(-a)=0$

Q. 24. $f'(0)=\lim_{n\to\infty} nf\left(\frac{1}{n}\right)$ and $f(0) = 0.$ Using this find

$$\lim_{n\to\infty}\left((n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-n\right),\left|\cos^{-1}\frac{1}{n}\right|<\frac{\pi}{2}$$

Ans. $\frac{\pi-2}{\pi}$

Solution. To find,

$$\lim_{n\to\infty}\left[(n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-n\right]$$

$$=\lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-1\right]=\lim_{n\to\infty} nf\left(\frac{1}{n}\right)$$

where $f(x)=\left[(1+x)\frac{2}{\pi}\cos^{-1}x-1\right]$ such that

$$f(0)=\left[(1+0)\frac{2}{\pi}\cos^{-1}0-1\right]=\frac{2}{\pi}.\frac{\pi}{2}-1=0$$

$\therefore$ Using given relation as $\lim_{n\to\infty} nf\left(\frac{1}{n}\right)=f'(0)$

then given limit becomes

$$= f'(0) = \frac{d}{dx}\left[(1+x)\frac{2}{\pi}\cos^{-1}x - 1\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\cos^{-1}x - \frac{1-x}{\sqrt{1-x^2}}\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 1\right] = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}.$$

Q. 25. If $|c| \le \frac{1}{2}$ and $f(x)$ **and f(x) is a differentiable function at x = 0 given**

$$\text{by } f(x) = \begin{cases} b\sin^{-1}\left(\frac{c+x}{2}\right) & , \quad -\frac{1}{2} < x < 0 \\ \frac{1}{2} & , \quad x = 0 \\ \frac{e^{ax/2}-1}{x} & , \quad 0 < x < \frac{1}{2} \end{cases}.$$

Find the value of 'a' and prove that 64 b^2 = 4 – c^2

Ans. 1

Solution. Given that, f (x) is differentiable at x = 0.

Hence, f (x) will also be continuous at x = 0

$$\Rightarrow \lim_{h\to 0} f(0+h) = f(0) \Rightarrow \lim_{h\to 0} \frac{e^{\frac{ah}{2}}-1}{h} = \frac{1}{2}$$

$$\Rightarrow \lim_{h\to 0} \frac{e^{\frac{ah}{2}}-1}{\frac{ah}{2}} \times \frac{a}{2} = \frac{1}{2} \Rightarrow a = 1$$

Also differentiability of f (x) at x = 0, gives

$$Lf'(0) = Rf'(0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{b\sin^{-1}\left(\frac{c-h}{2}\right) - \frac{1}{2}}{-h} = \lim_{h\to 0} \frac{e^{\frac{ah}{2}} - 1 - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2e^{\frac{ah}{2}} - 2 - h}{2h^2} \quad [\text{form } \frac{0}{0}]$$

$$\lim_{h\to 0} \frac{\frac{b}{\sqrt{1-\left(\frac{c-h}{2}\right)^2}} \cdot \left(-\frac{1}{2}\right)}{-1} \text{ [Using L Hospital's rule]}$$

$$= \lim_{h\to 0} \frac{2.e^{\frac{ah}{2}} \cdot \frac{a}{2} - 1}{4h} = \lim_{h\to 0} \frac{e^{\frac{h}{2}} - 1}{8\left(\frac{h}{2}\right)} \text{ [Putting } a = 1]$$

$$\Rightarrow \frac{b}{\sqrt{1-\frac{c^2}{4}}} = \frac{1}{8} \Rightarrow 4b = \sqrt{1-\frac{c^2}{4}} \Rightarrow 16b^2 = \frac{4-c^2}{4}$$

$$\Rightarrow 64b^2 = 4 - c^2 \quad \text{Hence proved.}$$

Q. 26. If f (x - y) = f (x) ×g (y) - f (y)×g (x) and g (x -y) = g (x) × g (y)- f (x)×f (y) for all x, y ∈ R . If right hand derivative at x = 0 exists for f(x). Find derivative of g(x) at x = 0

Ans. 0

Solution. Given that,

f (x – y) = f (x). g (y) – f (y) . g (x) ...(i)

g (x – y) = g (x). g (y) + f (x) f (y) ...(ii)

In eqn. (i), putting x = y, we get

f (0) = f (x) g (x) – f (x) g (x) Þ f (0) = 0

Putting y = 0, in eqn. (i), we get

f (x) = f (x) g (0) – f (0) g (x)

⇒ f (x) = f (x) g (0) [using f (0) = 0]

⇒ g (0) = 1

Putting x = y in eqn. (ii), we get

g (0) = g (x) g (x) + f (x) f (x)

⇒ 1 = $[g(x)]^2 + [f(x)]^2$ [using g (0) = 1]

⇒ [g (x)]2 = 1 – $[f(x)]^2$ ...(iii)

clearly g (x) will be differentiable only if f (x) is differentiable.

∴ First we will check the differentiability of f (x) Given that Rf ' (0) exists

$$\text{i.e., } \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \text{ exists}$$
$$\text{i.e., } \lim_{h\to 0}\frac{f(0)g(-h)-f(-h)g(0)}{h} \text{ exists}$$
$$\text{i.e., } \lim_{h\to 0}\frac{-f(-h)}{h} \text{ exists } (\text{using } f(0)=0 \text{ and } g(0)=1)$$

Which can be written as,

$$\lim_{h\to 0}\frac{f(0)-f(-h)}{-h}=Lf'(0)$$
$$\Rightarrow\ Lf'(0)=Rf'(0)$$
$$\therefore\ f \text{ is differentiable, at } x=0$$

Differentiating equation (iii), we get

$$2g(x).\,g'(x)=-2f(x).\,f'(x)$$
$$\text{For } x=0$$
$$\Rightarrow\ g(0).\,g'(0)=-f(0)f'(0)$$
$$\Rightarrow\ g'(0)=0 \qquad [\text{Using } f(0)=0 \text{ and } g(0)=1]$$

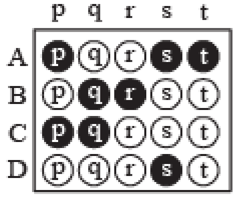
Match the following Question

Match the following

DIRECTIONS (Q. 1 and 2) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column II are labelled p, q, r, s and t.

Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II.

The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example: If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book.

	Column I		*Column II*
(A)	$\sin(\pi[x])$	(p)	differentiable everywhere
(B)	$\sin(\pi(x-[x])$	(q)	nowhere differentiable
		(r)	not differentiable at 1 and – 1

Ans. (A) - p, (B) - r

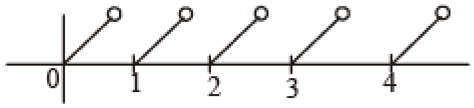
Solution. (A) $\sin(\pi[x]) = 0,\ \forall\ x \in R$

∴ Differentiable everywhere.

∴ (A) → (p)

(B) sin (p (x – [x])) = f (x)
We know that

$$x-[x]\begin{cases} = x, \text{if } 0 \le x < 1 \\ = x-1, \text{if } 1 \le x < 2 \\ = x-2, \text{if } 2 \le x < 3 \end{cases}$$



It’s graph is, as shown in figure which is discontinuous at $\forall\ x \in z.$ Clearly x – [x] and

hence sin (p (x – [x])) is not differentiable $\forall\ x \in z.$

(B) → r

Q. 2. In the following [x] denotes the greatest integer less than or equal to x. Match the functions in Column I with the properties in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

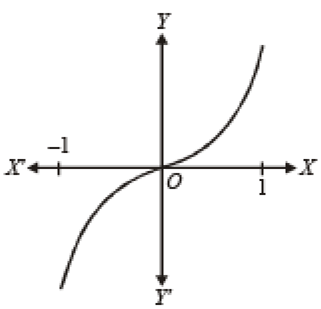
Column I	*Column II*
(A) $x\|x\|$	(p) continuous in (–1, 1)
(B) $\sqrt{\|x\|}$	(q) differentiable in (–1, 1)
(C) $x+[x]$	(r) strictly increasing in (–1, 1)
(D) $\|x-1\|+\|x+1\|$	(s) not differentiable at least at one point in (–1, 1)

Ans. (A) - p, q, r ; (B) - p, s ; (C) - r, s ; (D) - p, q

Solution.

(A) $y = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

Graph is as follows :



From graph y = x | x | is continuous in (– 1, 1) (p)

differentiable in (– 1, 1) (q)

Strictly increasing in (–1, 1). (r)

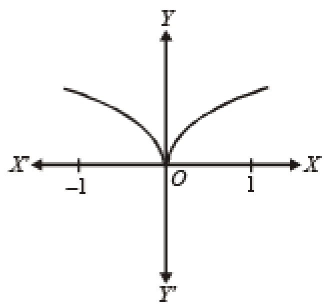
(B)

$$y = \sqrt{|x|} = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

{where y can take only + ve values}

and $y^2 = x$, $x \geq 0$

∴ Graph is as follows :

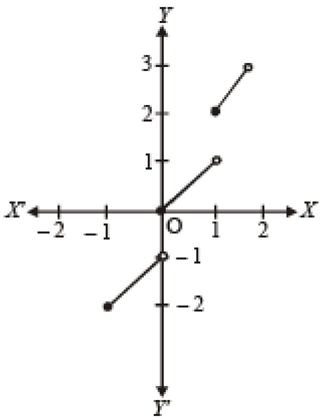


From graph $y = \sqrt{|x|}$ is continuous in (– 1, 1) (p) not differentiable at x = 0 (s)

(C) NOTE THIS STEP

$$y = x + [x] = \begin{cases} - & - - \\ x-1, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ x+1, & 1 \leq x < 2 \\ - & - - \end{cases}$$

∴ Graph of y = x + [x] is as follows :

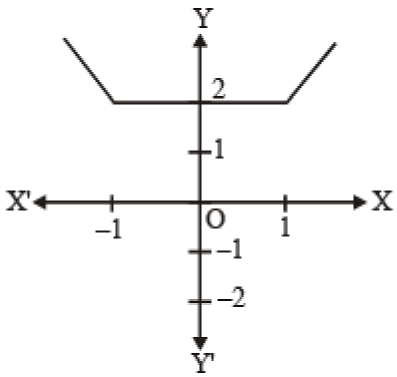


From graph, y = x + [x] is neither con tin uous, nor differentiable at x = 0 and hence in (– 1, 1). (s)

Also it is strictly increasing in (– 1, 1) (r)

(D) $y = |x-1| + |x+1| = \begin{cases} -2x, & x < -1 \\ 2, & -1 \le x < 1 \\ 2x, & x \ge 1 \end{cases}$

Graph of function is as follows :



From graph, y = f (x) is continuous (p) and differentiable (q) in (– 1, 1) but not strictly increasing in (– 1, 1).

DIRECTIONS (Q. 3) : Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Q. 3. Let $f_1 : R \to R$, $f_2 : [0, \infty) \to R$, $f_3 : R \to R$ and $f_4 : R \to [0,\infty)$ be defined

by $f_1(x)=\begin{cases}|x| & \text{if } x<0,\\ e^x & \text{if } x\ge 0;\end{cases}$

$f_2(x)=x^2$; $f_3(x)=\begin{cases}\sin x & \text{if } x<0,\\ x & \text{if } x\ge 0;\end{cases}$ and $f_4(x)=\begin{cases}f_2(f_1(x)) & \text{if } x<0,\\ f_2(f_1(x))-1 & \text{if } x\ge 0.\end{cases}$

	List-I		List-II
P.	f_4 is	**1.**	Onto but not one-one
Q.	f_3 is	**2.**	Neither continuous nor one-one
R.	$f_2 o f_1$ is	**3.**	Differentiable but not one-one
S.	f_2 is	**4.**	Continuous and one-one

	P	Q	R	S
(a)	3	1	4	2
(b)	1	3	4	2
(c)	3	1	2	4
(d)	1	3	2	4

Ans. (d)

Solution:

$$P(1):\ f_4(x)=\begin{cases}x^2, & x<0\\ e^{2x}-1, & x\ge 0\end{cases}$$

Range of $f_4=[0,\infty)$

$\therefore f_4$ is onto

From graph f_4 is not one one.

$$Q(3):\ f_3(x)=\begin{cases}\sin x, & x<0\\ x, & x\ge 0\end{cases}$$

From graph f is differentiable but not one.

$$R(2):\ f_2 o f_1(x)=\begin{cases}x^2, & x<0\\ e^{2x}, & x\ge 0\end{cases}$$

From graph f_2 of$_1$ is neither continuous nor one.

$$S(4):\ f_2(x)=x^2, x\in[0,\infty)$$

Integer Value Correct Type

Q. 1. Let f : [1, ∞) → [2, ∞) be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ **for all x $\geq$ 1 , then the value of f (2) is**

Ans. 6

Solution.

$$6\int_1^x f(t)dt = 3xf(x) - x^3$$

Differentiating, we get 6f (x) = 3 f (x) + 3xf '(x) – $3x^2$

$$\Rightarrow f'(x) - \frac{1}{x}f(x) = x$$

$$\text{I.F.} = \frac{1}{x}$$

$$\therefore \text{Solution is } f(x).\ \frac{1}{x} = \int 1.dx = x + c$$

∴ f (x) = x^2 + cx

But f (1) = 2 ⇒ c = 1

∴ f (x) = x^2 + x

Hence f (2) = 4 + 2 = 6

Note : Putting x = 1 in given integral equation, we get

$$f(1) = \frac{1}{3} \text{ while given } f(1) = 2.$$

∴ Data given in the question is inconsistent.

Q. 2. The largest value of non-negative integer a for

which $\lim_{x\to1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is

Ans. 2

Solution.

$$\lim_{x\to1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$$

$$\Rightarrow \lim_{x\to1}\left\{\frac{a(1-x)+\sin(x-1)}{(x-1)+\sin(x-1)}\right\}^{1+\sqrt{x}}$$

$$\Rightarrow \lim_{x\to1}\left\{\frac{-a+\frac{\sin(x-1)}{x-1}}{1+\frac{\sin(x-1)}{x-1}}\right\}^{1+\sqrt{x}} \Rightarrow \left(\frac{-a+1}{2}\right)^2=\frac{1}{4}$$

$$\Rightarrow a=0 \text{ or } 2$$

$\therefore$ Largest value of a is 2.

Q. 3. Let f : R →R and g : R → R be respectively given by f (x) = | x | + 1 and g(x) = x^2 + 1. Define h : R → R by

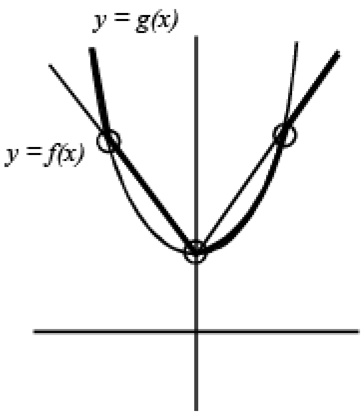
$$h(x)=\begin{cases}\max\ \{f(x),g(x)\} & \text{if } x\le 0,\\ \min\ \{f(x),g(x)\} & \text{if } x>0.\end{cases}$$

The number of points at which h(x) is not differentiable is

Ans. 3

Solution. $f(x)=|x|+1=\begin{cases}x+1, & x\ge 0\\ -x+1, & x<0\end{cases}$

g(x) = x^2 + 1



From graph there are 3 points at which h(x) is not differentiable.

Q. 4. Let m and n be two positive integers greater than 1.

If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$ **then the value of** $\frac{m}{n}$ is

Ans. 2

Solution.

$$\lim_{\alpha\to 0}\frac{e^{\cos\alpha^n}-e}{\alpha^m}=\frac{-e}{2}$$

$$\Rightarrow \lim_{\alpha\to 0}\frac{e\left[e^{\cos\alpha^n-1}-1\right]}{\cos\alpha^n-1}\times\frac{\cos\alpha^n-1}{\alpha^m}=\frac{-e}{2}$$

$$\Rightarrow e\lim_{\alpha\to 0}\frac{-2\sin^2\frac{\alpha^n}{2}}{\left(\frac{\alpha^n}{2}\right)^2}\times\frac{\left(\frac{\alpha^n}{2}\right)^2}{\alpha^m}=\frac{-e}{2}$$

$$\Rightarrow \frac{-e}{2}\alpha^{2n-m}=\frac{-e}{2}\text{ or }\alpha^{2n-m}=1$$

$$\Rightarrow 2n-m=0\Rightarrow\frac{m}{n}=2$$

Q. 5. Let $\alpha, \beta \in \mathbb{R}$ **be such that** $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1.$ **Then 6 (α + β) equals.**

Ans. 7

Solution.

$$\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x-\sin x}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{\alpha x-\sin x}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{\alpha x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\infty\right)}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{(\alpha-1)x+\frac{x^3}{3!}-\frac{x^5}{5!}+\ldots\infty}=1$$

For above to be possible, we should have

$$\alpha-1=0 \text{ and } \beta=\frac{1}{3!}$$

$$\Rightarrow \alpha=1 \text{ and } \beta=\frac{1}{6}$$

$$\therefore 6(\alpha+\beta)=6\left(1+\frac{1}{6}\right)=7$$